Improving Mean Forecast Using Density Forecast: A Functional Regression Approach

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Why We Need Point Forecasts?

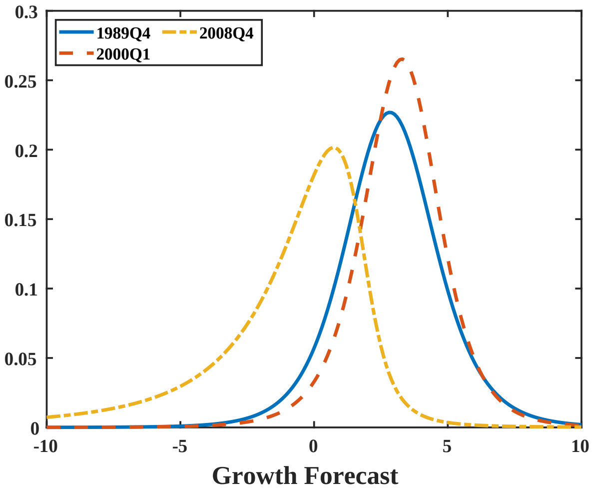
"My approach to this problem [uncertainty over model selection] while on the Federal Reserve Board was relatively simple: (...) simulate a policy on as many of these models as possible, throw out the outlier(s), and average the rest to get a point estimate of a dynamic multiplier path." (Blinder, 1999)

- Forward-looking decision-making, anchoring expectations, communication, and policy rules (e.g., Taylor Rule) often depend on single estimates
- However, they need to be **reliable** even under the presence of measurement errors, model misspecification, structural changes, uncertainty, among others, which is difficult to achieve

What About Density Forecasts?

- Density forecast was mainly used for uncertainty quantification
- Recently, it is also used to measure downside risks in economic growth (Adrian et al., 2019)
- **Density forecast could provide predictive content:**
  - Asymmetries in distributions of outcomes (Adrian et al., 2019)
  - Regimes, structural breaks (Chauvet and Potter, 2013)
  - Stochastic volatility (Brownlees and Souza, 2021; Carriero et al., 2024a)
  - Multimodalities in predictive distributions (Mitchell et al., 2024)
- Point and density forecasts do not necessarily have overlapping predictive content
  - Prepared by different teams, with different model/surveys, and information sets

U.S. GDP Growth Density Forecasts One-Year Ahead Conditional on Past GDP Growth and Financial Conditions



Research Question

Can the predictive content in density forecasts be exploited to improve the accuracy of point (mean) forecasts?

Related Literature

Growth-at-risk: (Adrian et al., 2019, 2022; Amburgey and McCracken, 2023)

Forecasting and modeling with many predictors (Stock and Watson, 2002, 2006, 2016)

Quantile and density forecasting in Empirical Macroeconomics (Brownlees and Souza, 2021; Carriero et al., 2024a,b; Corradi et al., 2023)

Functional time series (Chang et al., 2016, 2021; Park and Qian, 2012)

Our Paper

Proposes a way to improve point (mean) forecast using density forecast with functional regressions

Contributions of our method:

1. Offers a feasible and rigorous way of aligning point forecasts with underlying distributional risks or uncertainties
2. Exploits the predictive content of density forecast in an effective way by construction
3. Allows interpretation about which components of the density forecasts provide improvement

Some Notation and Background

- H: Hilbert Space of square integrable functions whose integral vanishes

$$H = \left\{ v \,\middle|\, \int v(r)dr = 0, \int v^2(r)dr < \infty \right\}$$

- For u, $v \in H$ the norm and inner product are

$$\langle u, v \rangle = \int u(r)v(r)dr, \quad \|v\|^2 = \langle v, v \rangle = \int v^2(r)dr,$$

- **Riesz Representation Theorem:** Any linear functional $\ell : H \to \mathbb{R}$ of $\left(f_{t+h|t}\right)$ can be represented as $\ell\left(f_{t+h|t}\right) = \left\langle \beta, f_{t+h|t} \right\rangle$ for some $\beta \in H$

Some Notation and Background

- **Parseval's Identity:** Any $v \in H$ can be written as

$$v = \sum_{k=1}^{\infty} \langle u_k, v \rangle u_k$$

  using any countably infinite orthonormal basis (u_k). Each $u_k \in H$

- V: subspace of dimension K of H

- $(u)'(v)$: inner product in $\mathbb{R}^K$ equivalent to $\langle u, v \rangle$ in $V \subseteq H$ given appropriate mapping (isometry) and projection.

The Model

Our goal is to estimate the functional regression

$$(y_{t+h} - \overline{y}) = \alpha(m_{t+h|t} - \overline{m}) + \langle \beta, (f_{t+h|t} - \overline{f}) \rangle + \varepsilon_{t+h}$$

- (y_{t+h}): observed target variable at time $t + h$
- $(m_{t+h|t})$: point forecast conditional on information up to t
- $(f_{t+h|t})$: density forecast conditional on information up to t
- $\overline{y}$, $\overline{m}$, and $\overline{f}$ are sample averages

In $\langle \beta, (f_{t+h|t} - \overline{f}) \rangle$, β and $(f_{t+h|t} - \overline{f})$ are infinite-dimensional functional components which cannot be used directly in the estimation
It is equivalent to

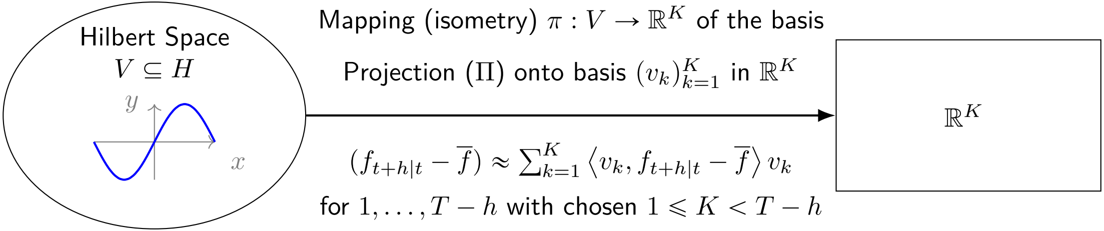
$$y_{t+h} = \tau + \alpha\, m_{t+h|t} + \langle \beta, f_{t+h|t} \rangle + \varepsilon_{t+h},$$

with $\tau = \bar{y} - \alpha\bar{m} - \langle \beta, \bar{f} \rangle$. We called this specification *unrestricted* (ur) Mean Factor Specifications

From Functional Observations to Finite-Dimensional Vectors

$\overline{\Gamma} = \frac{1}{T-h}\sum_{t=1}^{T-h}\left((f_{t+h|t} - \overline{f}) \otimes (f_{t+h|t} - \overline{f})\right)$ is the sample variance operator of $(f_{t+h|t})$

We use the *functional principal components* (FPCs) of $(f_{t+h|t})$, which are the eigenfunctions of $\overline{\Gamma}$, which form a countable basis denoted by $(v_k)_{k=1}^{T-h}$. They are ranked by proportion of variance explained. Let $V \subseteq H$ be of dimension K



$$\langle \beta, (f_{t+h|t} - \overline{f}) \rangle \approx \langle \beta, \Pi(f_{t+h|t} - \overline{f}) \rangle = (\beta)'(f_{t+h|t} - \overline{f})$$

All $f_{t+h|t}$ and v_k are functions discretized with M values for feasibility.

Estimation Procedure

Then, we go from a functional regression to a standard regression

$$(y_{t+h} - \overline{y}) = \alpha\,(m_{t+h|t} - \overline{m}) + \left\langle \beta,\, (f_{t+h|t} - \overline{f}) \right\rangle + \varepsilon_{t+h}$$

$$\Downarrow$$

$$(y_{t+h} - \overline{y}) \approx \alpha\,(m_{t+h|t} - \overline{m}) + \llbracket\beta\rrbracket'\llbracket f_{t+h|t} - \overline{f}\rrbracket + \varepsilon_{t+h}$$

which can be estimated with OLS.

The functional regression coefficient β can be recovered by

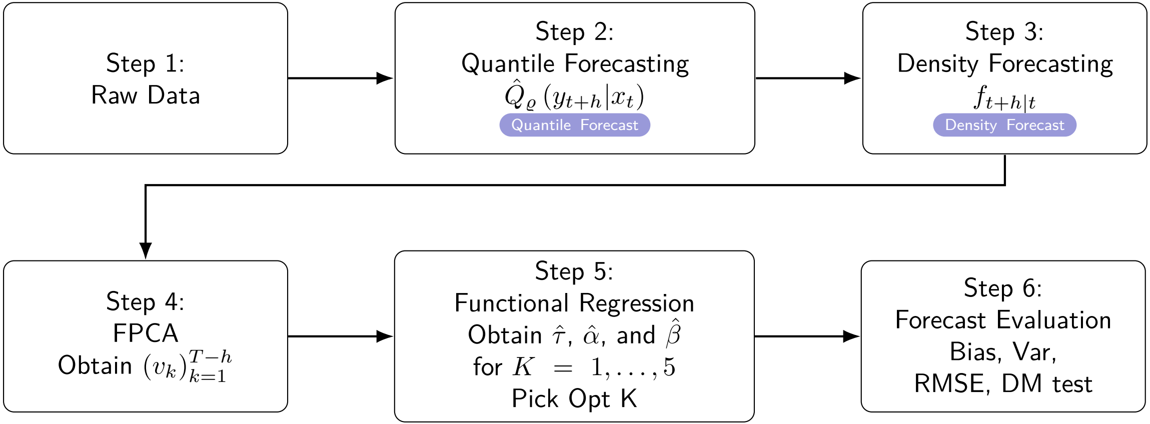
$$\hat{\beta} = \pi^{-1}(\widehat{\llbracket\beta\rrbracket})$$

One-Quarter-Ahead and One-Year-Ahead Forecasting of the U.S. Real GDP Growth Rate

Data

- Sample is 1973Q1-2024Q3
- We use revised and vintage data for the U.S. from FRED and ALFRED
- **Rolling Window:** First estimation sample 1973Q1-2003Q4 for $h = 1$, 1973Q1-2003Q1 for $h = 4$, forecast 2004Q1 in both. Out-of-sample (OOS) 2004Q1-2024Q3
- **Target variable (y_{t+h}):** quarterly GDP growth rate at $t + h$
  - *For $h = 1$:* annualized quarter-over-quarter rate
  - *For $h = 4$:* annualized year-over-year rate
- **Covariates (x_t) for density forecasts:** rates are always the most recent annualized quarter-over-quarter rate for both $h = 1$ and $h = 4$
  - GDP growth rate and NFCI at t
- **Point forecasts (m_t) for functional regression:** mean value of individual responses from the Survey of Professional Forecasters (SPF)
- **Density forecasts (f_t) for functional regression:** computed with quantile regression and fitting parametric t-skewed distribution using y_{t+h} and x_t

Modeling Procedure



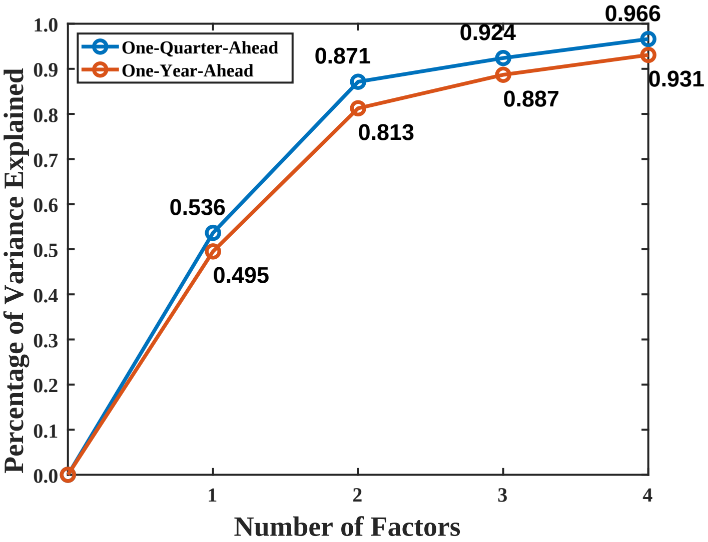
Modeling Procedure

- Pick Opt K with BIC using first rolling window
- Estimate quantile/density forecasting, FPCs, and functional regression in each rolling window
- Fitted value, $\hat{y}_{t+h}$ is a point forecast
- Forecast Evaluation: 83 observations OOS.
  - Diebold-Mariano (DM) test:

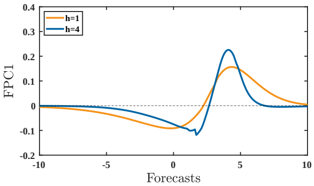
$$DM_{12} = \frac{\bar{d}_{12}}{\widehat{\sigma}_{\bar{d}_{12}}} \xrightarrow{d} N(0,1) \text{ under the null hypothesis } DM_{12} = 0$$

where $d_{12,t+h} = \varepsilon^2_{1,t+h} - \varepsilon^2_{2,t+h}$. $\varepsilon^2_{1,t+h}$: forecasting errors of our proposed functional regression. $\varepsilon^2_{2,t+h}$: forecasting errors of $m_{t+h|t}$. *Negative DM statistic means improvement of proposed functional regression over point forecast*

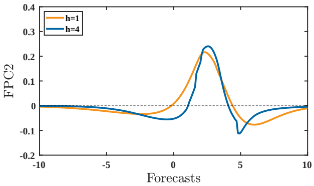
FPCA: GDP Growth Scree Plots. Revised Data. 1974Q1-2024Q3



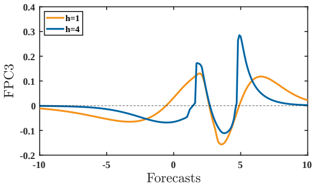
FPCA: GDP Growth FPC Estimates. Revised Data. 1974Q1-2024Q3



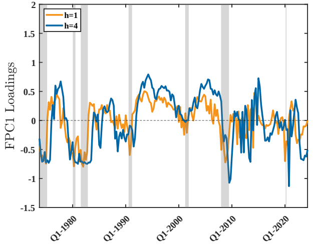
(a) First FPC



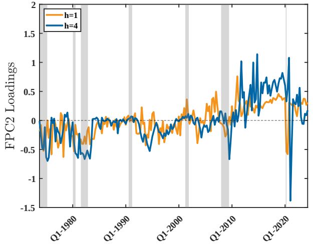
(b) Second FPC



(c) Third FPC



(d) First FPC Loadings



(e) Second FPC Loadings

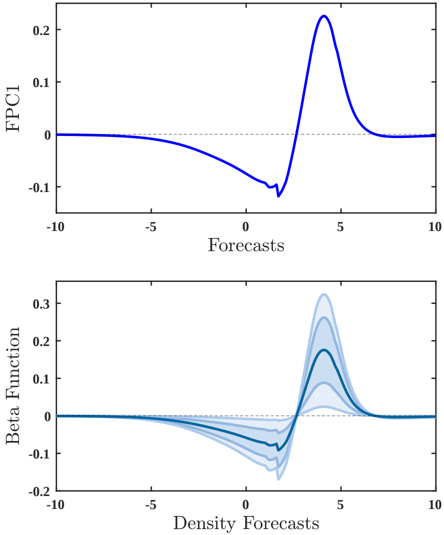
(f) Third FPC Loadings

Estimates: One-Year-Ahead GDP Growth. Revised Data. 1974Q1-2024Q3

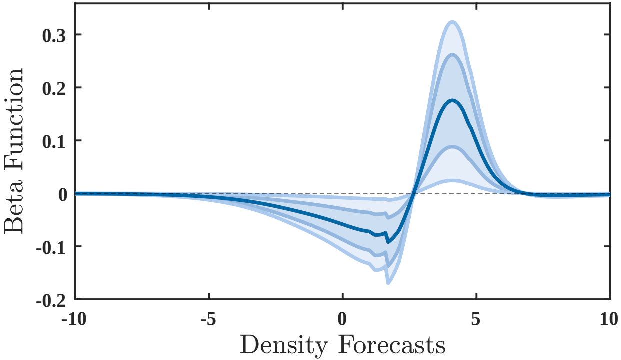
	ur
$m_{t+h\|t}$	0.3970
	[0.2244, 0.5793]
$f_{t+h\|t}^{FPC_1}$	0.7779
	[0.1077, 1.4347]
K	1
Observations	203

$$\hat{\beta} = \pi^{-1}(\widehat{[\beta]}) = 0.7779 \times v_1$$

More Results | $rr: \hat{\beta}$ | $u\mu r: \hat{\beta}_c$ | $r\mu r: \hat{\beta}_c$



Estimated $\hat{\beta}$ Function from ur Model



$\hat{\beta}$ shows the following:

- $f_{t+h|t}$ pushes down $m_{t+h|t}$ when the forecasts are small, less than 2.5%
- $f_{t+h|t}$ pushes up $m_{t+h|t}$ when the forecasts are high, over 2.5%
- This suggests that the point forecast from the SPF is **conservative**

OOS Forecast Performance: GDP Growth, Vintage Data, 2004Q1-2024Q3

Horizon	Specification	Relative to SPF (in denominator) Bias	Var	RMSE	DM test	K
$h = 1$	ur	1.1693	1.3313	1.1541	2.0423**	2
$h = 4$	ur	0.2551	0.9657	0.6640	-3.8618***	2

More Results

Conclusions

- The SPF's point forecast **can be improved** when used jointly with density forecast for one-year-ahead GDP growth
- Unrestricted specifications perform better for GDP growth
- The improvement comes from **exploiting predictive content from the density forecast related to recession and expansion periods**
- The results indicate that the SPF's consensus (average) forecast tends to be **conservative**
- *Improvement is case-specific since it depends on the quality of the point and the density forecasts as well as overlapping predictive content*

Thank you!

Questions or comments?

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Appendix

In Which Way Does The Density Forecast Provide Improvement? Back

1. The density forecast could help without modifications

2. We can also separate the mean factor from the rest of the density

Instead of the leading FPC, set the first factor to be $u_m(r) = r$ meaning that

$$\langle u_m, f_{t+h|t} \rangle = \int r f_{t+h|t}(r) dr = \mu(f_{t+h|t})$$

We call u_m the *mean factor*

Then, we define a *centered* density forecast $(f_{c,t+h|t})$ as

$$f_{c,t+h|t} = f_{t+h|t}(r + \mu(f_{t+h|t}))$$

We could lose variance explained this way, but it let us know which part of the density helps

Functional Regression Specifications Back

1. Leading FPC first:

Functional Regression Specifications Back

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   ur. *Unrestricted*:

$$y_{t+h} = \tau + \alpha m_{t+h|t} + \langle \beta, f_{t+h|t} \rangle + \varepsilon_{t+h}$$

Functional Regression Specifications Back

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   rr. *Restricted*:

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Functional Regression Specifications

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   rr. *Restricted*:

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Functional Regression Specifications Back

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$$y_{t+h} = \tau + \alpha m_{t+h|t} + \beta_m \mu(f_{t+h|t}) + \left\langle \beta_c, f_{c,t+h|t} \right\rangle + \varepsilon_{t+h}$$

Functional Regression Specifications Back

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Parametric Quantile Forecast With Few Covariates

Quantile forecast:

$$\hat{Q}_{\varrho}\left(y_{t+h}|x_t\right) = x_t'\hat{\gamma}_{\varrho}$$

where

$$\hat{\gamma}_{\varrho} = \underset{\gamma_{\varrho}\in\mathbb{R}^M}{\arg\min} \sum_{t=1}^{T-h} \left(\varrho \cdot 1_{(y_{t+h} \geqslant x_t'\gamma_{\varrho})} \left|y_{t+h} - x_t'\gamma_{\varrho}\right| + (1-\varrho) \cdot 1_{(y_{t+h} < x_t'\gamma_{\varrho})} \left|y_{t+h} - x_t'\gamma_{\varrho}\right|\right)$$

$1_{(\cdot)}$ denotes the indicator function, y_{t+h} is the target variable in $t+h$, x_t are covariates in t, and ϱ is the respective quantile index

Parametric Quantile Forecast With Data-Rich Environment

LASSO Quantile forecast: Modify the LASSO penalty (Belloni and Chernozhukov, 2011)

$$\hat{\gamma}_\varrho = \arg\min_{\gamma_\varrho \in \mathbb{R}^M} \sum_{t=1}^{T-h} \left(\varrho \cdot 1_{(y_{t+h} \geqslant x_t' \gamma_\varrho)} \left| y_{t+h} - x_t' \gamma_\varrho \right| + (1-\varrho) \cdot 1_{(y_{t+h} < x_t' \gamma_\varrho)} \left| y_{t+h} - x_t' \gamma_\varrho \right| \right) + \frac{\lambda \sqrt{\varrho(1-\varrho)}}{T-h} \sum_{j=1}^{M} |\gamma_{\varrho,j}|$$

with $\{\gamma_{\varrho,j}\}_{j=1}^{M}$ being elements of the coefficient vector γ_ϱ for each quantile index, ϱ.

Lasso Selection

Parametric Quantile Forecast With Data-Rich Environment

Belloni and Chernozhukov (2011) suggest setting λ as

$$\lambda = c \cdot \Lambda(1 - \alpha | X),$$

where X is the matrix of covariates, $\Lambda(1 - \alpha | X)$ is the $(1 - \alpha) = 0.95$-quantile of the random variable Λ conditional on X, and $c = 2$ is a constant. We simulate a 1,000 draws from Λ to obtain λ, with

$$\Lambda = M \sup_{\varrho \in \mathcal{P}} \max_{1 \leqslant j \leqslant M} \left| \mathbb{E}_M \left[\frac{x_{ij}(\varrho - 1\{\varrho_i \leqslant \varrho\})}{\hat{\sigma}_j \sqrt{\varrho(1-\varrho)}} \right] \right|,$$

where M is total number of covariates, $u_1, \ldots, u_M$ are i.i.d. uniform $(0, 1)$ random variables, independently distributed from the covariates, $x_1, \ldots, x_M$, and $\hat{\sigma}_j = \mathbb{E}_M[x_{ij}^2]$ is an estimate of the covariance matrix of the covariates.

Nonparametric Quantile Forecast With Data-Rich Environment

Quantile Generalized Random Forest forecast:

$$\hat{Q}_{y_{t+h}|x_t}(\varrho|x_t) = \hat{g}_\varrho(x_t)$$

where $g_\varrho(\cdot)$ is an unknown function and $\hat{g}_\varrho(\cdot)$ a consistent estimator.

We use the generalized random forest proposed by Athey et al. (2019).

- Allows for locally weighted estimation with gradient-based quantile regression trees.
- Create child nodes by splitting observations above and below the ϱth quantile in each parent node
- Instead, the standard random forest uses trees that decide on each split using the mean squared error for expected value.

Quantile Generalized Random Forest forecast:

We estimate local moment conditions of the form

$$\mathbb{E}[\psi_\varrho(Y_{t+h})|X_t = x_t] = 0$$

where

$$\psi_\varrho(Y_{t+h}) = \varrho \cdot 1_{\{y_{t+h} \geqslant g_\varrho(x_t)\}} \left|y_{t+h} - g_\varrho(x_t)\right| + (1-\varrho) \cdot 1_{\{y_{t+h} < g_\varrho(x_t)\}} \left|y_{t+h} - g_\varrho(x_t)\right|$$

To obtain $\hat{g}_\varrho(x_t)$.

Computation of Quantile Forecasts

- **Parametric quantiles with few covariates and a data-rich environment with quantile LASSO:**

  We compute few quantiles ($\varrho \in \{0.05, 0.25, 0.50, 0.75, 0.95\}$)

- **Parametric quantiles with few covariates and nonparametric quantiles in a data-rich environment with generalized random forest:**

  We compute many quantiles (near 400)

  - Estimates suffer from quantile crossing. We rearrange quantiles using method of Chernozhukov et al. (2010)

Fixing Quantile Crossing

Consider a possibly non-monotone function $\varrho \rightarrow \hat{Q}_{y_{t+h}|x_t}(\varrho|x_t)$

We can transform it into a monotone function, $\varrho \rightarrow \hat{Q}^*_{y_{t+h}|x_t}(\varrho|x_t)$ using quantile bootstrap or rearrangement (Chernozhukov et al., 2010; Koenker, 1994)

Consider a random variable

$$Z_{y_{t+h}|x_t} := \hat{Q}_{y_{t+h}|x_t}(P|x_t),$$

where $P \sim$ Uniform($\mathcal{P}$) with $\mathcal{P} = (0, 1)$, and take its quantile function denoted by $\varrho \rightarrow \hat{Q}^*_{y_{t+h}|x_t}(\varrho|x_t)$ instead of the original function $\varrho \rightarrow \hat{Q}_{y_{t+h}|x_t}(\varrho|x_t)$

For each period t and given the forecasting horizon h, we draw a 1,000 random points for P and obtain the respective values for $Z_{y_{t+h}|x_t}$ and $\hat{Q}^*_{y_{t+h}|x_t}(\varrho|x_t)$ Back

Parametric Computation of Density Forecast ($f_{t+h|t}$)

We use $\hat{Q}_{y_{t+h}|x_t}(u|x_t)$ to fit a skewed t-distribution:

$$f(y;\mu,\sigma,\alpha,\nu) = \frac{2}{\sigma} t\left(\frac{y-\mu}{\sigma};\nu\right) T\left(\alpha \frac{y-\mu}{\sigma}\sqrt{\frac{\nu+1}{\nu+\left(\frac{y-\mu}{\sigma}\right)^2}};\nu+1\right)$$

where $t(\cdot)$ and $T(\cdot)$ denote the PDF and the CDF of the Student t-distribution, respectively. μ is the location, σ is the scale, ν is the fatness, and α is the shape.

Parametric Computation of Density Forecast ($f_{t+t|t}$)

For each quarter, t, and forecasting horizon, h, the fitting process requires finding $\{\hat{\mu}_{t+h}, \hat{\sigma}_{t+h}, \hat{\alpha}_{t+h}, \hat{\nu}_{t+h}\}$ as follows:

$$\{\hat{\mu}_{t+h}, \hat{\sigma}_{t+h}, \hat{\alpha}_{t+h}, \hat{\nu}_{t+h}\} = \underset{\mu,\sigma,\alpha,\nu}{\arg\min} \sum_{\varrho} \left(\hat{Q}_{\varrho}\left(y_{t+h} \mid x_t\right) - F^{-1}(u; \mu, \sigma, \alpha, \nu) \right)^2$$

where $F^{-1}(u; \mu, \sigma, \alpha, \nu)$ is the quantile function of the skewed t-distribution.

The minimization matches the 5, 25, 75, and 95 percent quantiles.

Nonparametric Computation of Density Forecast ($f_{t+h|t}$)

Let the realizations of $Z_{y_{t+h}|x_t}$ be denoted as the vector $z_{y_{t+h}|x_t}$ and its entries as $\{z_{y_{t+h}|x_t,i}\}_{i=1}^{n}$ with $n = 1,000$ for fixed values of t and h.

We estimate the respective nonparametric density forecast with Kernel Density Estimation:

$$\hat{f}_{Y_{t+h}|X_t}(y_{t+h}|x_t) = \frac{1}{nh_{\mathcal{K}}} \sum_{i=1}^{n} \mathcal{K}\left(\frac{z_{y_{t+h}|x_t,i} - z_{y_{t+h}|x_t}}{h_{\mathcal{K}}}\right)$$

with $\hat{f}_{Y_{t+h}|X_t}(y_{t+h}|x_t)$ an estimate of the density forecast, $\mathcal{K}(\cdot)$ being an appropriate kernel function, and $h_{\mathcal{K}}$ the smoothing parameter, also called bandwidth.

- We use the Epanechnikov Kernel
- We compute $h_{\mathcal{K}}$ LSCV with the 1,000 realizations of $Z_{y_{t+h}|x_t}$
- Optimal $h_{\mathcal{K}}$ changes for each t and h

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Quantile Lasso Variable Selection: Overview

- Quantile Lasso regressions in data-rich setups do not outperform the baseline in OOS performance.
- However, analyzing selected variables across quantiles remains informative.
- Focus:
  - One-year-ahead GDP growth
  - One-quarter-ahead CPI inflation
- Variables selected at quantiles $\varrho = 0.05$, 0.25, 0.50, 0.75, 0.95 (used in fitting the skewed t-distribution).
- **Number of selected variables:**
  - GDP growth: 6, 8, 15, 12, 7
  - CPI inflation: 7, 11, 13, 11, 9

Selected Variables: GDP Growth

- **Lower quantiles** ($\varrho = 0.05, 0.25$):
  - Interest rate spreads (T1YFFM, T5YFFM)
  - Financial conditions (ANFCI, NFCIRISK)
  - Housing supply and investment (MSACSR, PRFI)
- Interpretation: Weak financial conditions, interest rates spreads, and housing signal downside risks

- **Middle to upper quantiles** ($\varrho = 0.50, 0.75, 0.95$):
  - Labor market: average weekly hours (AWHNONAG), compensation (RCPHBS)
  - Sector-specific compensation: nondurable (CES3200000008), financial (CES5500000008)
- Interpretation: Stronger labor markets and sectoral compensation are associated with robust GDP growth

Selected Variables: CPI Inflation

- **Lower quantiles** ($\varrho = 0.05, 0.25$):
  - Capacity utilization (CAPUTLGMFNS), industrial energy production (IPB51220SQ)
  - Labor hours (AWHNONAG)
- Interpretation: Efficiency and low production costs are associated with low inflation

- **Middle to upper quantiles** ($\varrho = 0.50, 0.75, 0.95$):
  - Inflation expectations: MICH, EXPINF1YR
  - Core inflation: CPI less food (CPIULFSL), PCE excluding food, energy, and housing (IA001176M)
  - Wages (CES6500000008), producer prices (WPSFD49207)
- Interpretation: Cost-push and expectations play larger roles in high-inflation scenarios.

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Estimates: One-Year-Ahead GDP Growth. Revised Data. 1974Q1-2024Q3

	ur	rr	$u\mu r$	$r\mu r$
$m_{t+h\|t}$	0.3970		0.3229	
	[0.2244, 0.5793]		[0.1629, 0.4738]	
$\mu(f_{t+h\|t})$			0.8052	-0.1570
			[0.2706, 1.4070]	[-1.0784, 0.8265]
$f^{FPC_1}_{t+h\|t}$	0.7779	-0.7052		
	[0.1077, 1.4347]	[-1.3676, -0.0537]		
$f^{FPC_2}_{t+h\|t}$		-0.9402		
		[-1.7028, -0.1522]		
$f^{FPC_3}_{t+h\|t}$		1.0630		
		[-0.3656, 2.5533]		
$f^{FPC_4}_{t+h\|t}$		0.5839		
		[-1.3304, 2.4886]		
$f^{FPC_1}_{c,t+h\|t}$			-0.4940	-0.4323
			[-1.5585, 0.4915]	[-2.1733, 1.2227]
$f^{FPC_2}_{c,t+h\|t}$				-0.8766
				[-1.6436, -0.0999]
$f^{FPC_3}_{c,t+h\|t}$				1.4433
				[-0.9872, 3.7375]
K	1	4	1	3
Observations	203	203	203	203

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Estimated $\hat{\beta}$ Function from rr Model For One-Year-Ahead GDP Growth

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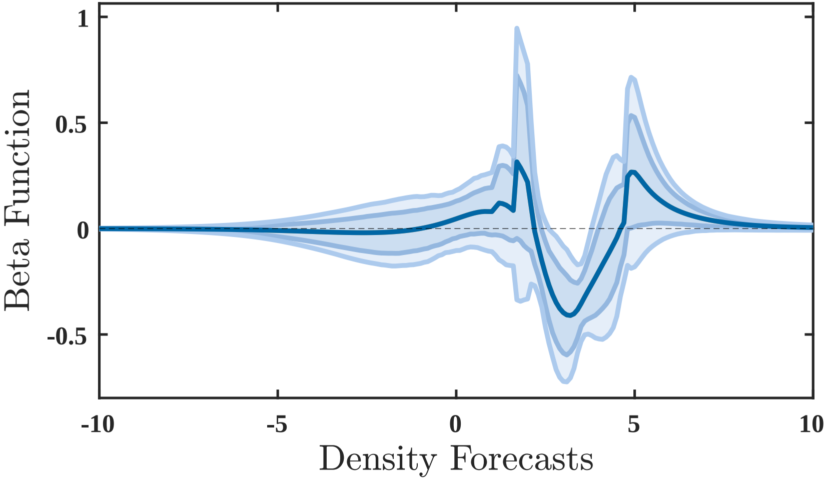

Estimated $\hat{\beta}_c$ Function from $u\mu r$ Model For One-Year-Ahead GDP Growth

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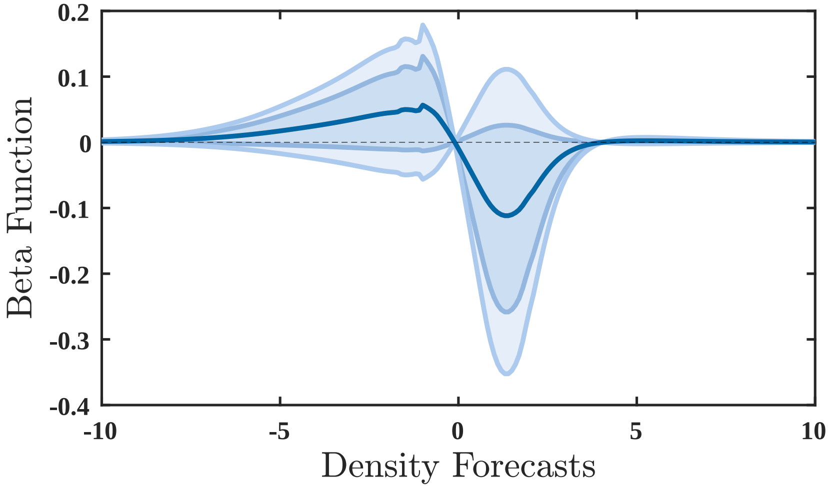

Estimated $\hat{\beta}_c$ Function from $r\mu r$ Model For One-Year-Ahead GDP Growth

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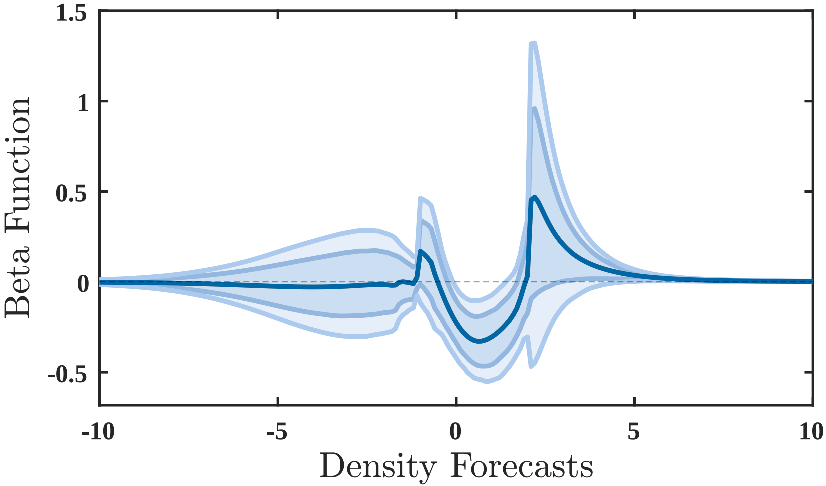

OOS Forecast Performance: GDP Growth, Vintage Data, 2004Q1-2024Q3

		Relative to SPF (in denominator)				
Horizon	Specification	Bias	Var	RMSE	DM test	K
$h = 1$	ur	1.1693	1.3313	1.1541	2.0423**	2
	rr	1.1717	1.1407	1.0696	1.8408*	1
	$u\mu r$	0.3716	1.4506	1.1964	1.2539	3
	$r\mu r$	0.5457	1.1100	1.0479	1.2731	3
$h = 4$	ur	0.2551	0.9657	0.6640	-3.8618***	2
	rr	0.2616	1.0456	0.6901	-3.3382***	1
	$u\mu r$	0.2108	1.6137	0.8362	-1.0245	1
	$r\mu r$	0.3029	1.0411	0.6985	-3.5177***	2

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